Buoyant plumes in a moist atmosphere

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SUMMARY

This paper describes a simple model which can be used to investigate the transport of water vapour by thermal plumes in the atmosphere. For an approximate treatment of these plumes, it is assumed (as in a previous paper) that the vertical velocity, temperature and specific humidity are constant across the ascending column, and that the inflow velocity due to mixing at the edge of the plume is proportional to the vertical velocity within the plume. The behaviour of the rising air is then investigated by means of equations representing the conservation of mass, momentum, heat and water vapour, and numerical solutions are obtained for representative cases.

It is shown that in a stably stratified atmosphere the plume air will become saturated if the source is sufficiently strong, and that the height of this saturation level can be determined in terms of dimensionless parameters representative of the source and the environment. Above the saturation level the analysis is continued by taking into account both vapour and liquid phases of water, and an approximate treatment is given for the behaviour of small plume clouds. It is found that there is a critical size for these; smaller clouds remain characteristic of the plume, but larger ones extend upwards to heights which do not depend on the previous parameters.

INTRODUCTION

An account has been given recently by Morton, Taylor & Turner (1956) of an approximate treatment for maintained buoyant plumes rising in a stratified environment under conditions such that the flow is turbulent. In this paper it was assumed that at each level the rate of entrainment at the edge of the plume is proportional to the mean vertical velocity at its centre. A similarity solution was then developed from equations representing the conservation of mass, momentum and weight deficiency (relative to the ambient fluid at some specified level, such as that of the plume source). The solution involves one undetermined constant, and for this a value was found from experiments in which a plume of dyed alcohol was released from the bottom of a tank containing stably stratified salt solution. Finally, the results of the theory were illustrated by a brief discussion of the ascent of thermal columns in the atmosphere. This application to atmospheric convection will be extended in the present work.

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When a convective plume develops from a source in a moist atmosphere, moisture will be entrained into the plume and carried up with the air. The resulting distribution of water vapour in the plume can be investigated by a simple extension of the methods of the earlier treatment, involving the introduction of a fourth conservation relation. Moreover, in the earth's atmosphere the concentration of moisture is generally greatest near the ground (or sea) and decreases steadily with height, and thus at each level the plume air will be moister than its surroundings. Since the absolute temperature of the air in the plume also decreases with height under normal conditions, this air may become saturated at some level; a cloud will then appear in the upper part of the plume if suitable nuclei are present for condensation.

This elementary model may possibly represent one mechanism for the formation of clouds, although certain reservations must be borne in mind if it is to be applied to the practical case. As the steady state rarely pertains to atmospheric phenomena such an approach can provide only a temporary description of a slowly varying condition. In addition, assuming that suitable plumes can form in the atmosphere, these rising columns will frequently be deflected by winds and under some conditions may lose their identity well below the heights predicted by the theory referred to above. However, it may be noted that condensation does actually appear to take place in air which has risen from lower levels. When atmospheric conditions are reasonably still, clouds are sometimes seen above plumes rising from the chimneys of large industrial plants: a recent photograph of such a case has been published by Scorer (1955), although a slightly different interpretation has been given. Clouds are also associated with fires in the cane-fields at Hawaii, and these will be discussed later. Thus, the transport of moisture vapour in buoyant plumes discussed in the present note may also provide one among many possible mechanisms for cloud formation, provided that weather conditions are reasonably calm and are not changing too rapidly.

The necessary steady or quasi-steady plumes may ascend from a variety of sources in the atmosphere. Rising columns can certainly be expected above artificial origins such as industrial plants and large fires, and possibly above cities. The effect of naturally occurring sources is more important, although rather more difficult to assess, but columns may be expected aboverock outcrops, bare earth, sand, or areas of sun-browned vegetation, when these are heated strongly by the sun. In each case the source will serve to 'organize' the heat transfer from some neighbourhood so that the plumes which grow may be much stronger than could be established by the source itself. In many cases these plumes will not penetrate to sufficiently great heights for condensation to be possible; they will then play an important but invisible part in the transfer of moisture in the atmosphere. Another possible source of plumes in the atmosphere has been suggested by Batchelor (1954) and Priestley (1955), who have shown that convective plumes can grow spontaneously from small disturbances in an unstably stratified layer. Although it seems likely that the effect of such a disturbance would be to start many plumes, and consequently to cause a general overturning of the layer, it is possible that heated air might continue to ascend above particularly vigorous regions in the layer and so give rise to plumes which penetrate into the neutral or stably stratified air above. These plumes will be at best quasi-steady because they have no fixed points of origin and because they are likely to exhaust the layer locally; but they may be of some importance in the transport of moisture upwards through the atmosphere.

The analysis given previously (Morton, Taylor & Turner 1956, hereafter referred to as paper I) was developed for incompressible fluids, but an extension was made to cover convection in the atmosphere. This was done by accepting the customary assumption that real convective motions in the earth's atmosphere, described by velocities, potential temperatures and potential densities, are formally equivalent to motions in a similar but incompressible region described by velocities, ordinary temperatures and densities. While this substitution of potential temperature for temperature in the equations representing convective motion of an incompressible fluid compensates for the adiabatic nature of the pressure changes, the modified equations can be regarded as valid only within regions of limited vertical extent and for small total variations in potential temperature. For example, it can be shown that if the modified equations are used to investigate convection in the lowest two kilometres of the atmosphere the error in the solution is likely to be as large as 10%. In spite of this, and although the effects due to radiation have been wholly neglected, it seems that the present type of analysis is likely to provide a more accurate description of some aspects of atmospheric convection than has been possible previously.

Steady plumes in a neutrally stable environment

In certain applications the environment will be of uniform density (or potential density for a gas), and this case will be considered first as there is an exact solution to the approximate equations. The motion above a small source of heat in a still atmosphere at uniform potential temperature may be referred to cylindrical polar coordinates (x, r), with the source at the origin and the x-axis directed vertically upwards. Let u be the vertical velocity, θ the potential temperature and ρ the density inside the plume; and let θ_0 and ρ_0 be the corresponding temperature and density in the ambient air. Further, take $\theta_1 = \theta_0(0)$ and $\rho_1 = \rho_0(0)$ as reference values for the system. Although the presence of water vapour in the atmosphere causes a small variation in the density and other physical parameters, this effect can be neglected, without introducing appreciable error, up to the height at which condensation begins. If it is now assumed that the velocity and buoyancy force are constant across the plume and zero outside it (i.e. a 'top hat' profile), the equations representing conservation of mass, momentum and weight deficiency can be written (see paper I, equation 3)

$$\frac{d}{dx}(b^2u) = 2\alpha bu, \tag{1}$$

$$\frac{d}{dx}(b^2u^2) = b^2\beta g(\theta-\theta_0), \tag{2}$$

$$\frac{d}{dx}[b^2u\beta g(\theta-\theta_0)] = 0, \tag{3}$$

where b is the horizontal radius of the plume, β the coefficient of cubical expansion, g the acceleration due to gravity, and α the proportionality constant relating the inflow velocity at the edge of the plume to the vertical velocity within the plume. For boundary conditions it may be assumed that at the source the radius and the flux of momentum in the plume are zero, and that heat is released at a known and constant rate. Integration of equation (3) gives

$$b^2u\beta g(\theta-\theta_0) = F/\pi, \tag{4}$$

where F is the flux of buoyancy. If ρ_1 is the density of the ambient air at source level and c_p is the specific heat at constant pressure, the steady rate of output of heat from the source is $\rho_1 c_p F/\beta g$. The full solution of equations (1), (2) and (3) with the given boundary conditions can be written

$$b = \frac{6\alpha}{5}x, \qquad u = \frac{5}{6\alpha}\left(\frac{9\alpha F}{10\pi}\right)^{1/3} x^{-1/3},$$

$$\beta g(\theta-\theta_0) = \frac{5F}{6\pi\alpha}\left(\frac{9\alpha F}{10\pi}\right)^{-1/3} x^{-5/3}. \tag{5}$$

In paper I, α was determined for a Gaussian profile since this fitted closely with the available experimental observations. The equivalent value of α for the 'top hat' profile can be found by matching the vertical flux of mass and momentum and the horizontal flow into the edge of the plume for the two profiles, and is found to be $\alpha = 0{\cdot}132$.

The distribution of moisture within the plume can now be investigated by introducing a fourth equation representing the conservation of water vapour:

$$\frac{d}{dx}(b^2qu) = 2bq_0\alpha u, \tag{6}$$

where q, the specific humidity in the plume, is the mass of water vapour in unit mass of moist air, and $q_0 = q_0(x)$ refers to the surroundings. When relations (5) are used, equation (6) reduces to the form

$$x\frac{dq}{dx} + \frac{5}{3}q = \frac{5}{3}q_0, \tag{7}$$

with the solution

$$q = \frac{5}{3}x^{-5/3}\left\{\text{constant} + \int x^{2/3}q_0(x)\,dx\right\}. \tag{8}$$

One additional boundary condition is needed to determine the single disposable constant, and this is provided by the known rate of release of moisture from the source, which is proportional to $E = [b^2uq]_{x=0}$.

Although $q_0(x)$ may have any given functional form, it is sufficient for the present purposes to assume a linear profile for the specific humidity in the atmosphere and to take

$$q_0(x) = A - Bx, \tag{9}$$

where A and B are profile constants. Then equations (8) and (9) lead to

$$q = A - \tfrac{5}{8}Bx + \frac{5E}{6\alpha}\left(\frac{9\alpha F}{10\pi}\right)^{-1/3} x^{-5/3}. \tag{10}$$

The first two terms of this expression depend on the moisture content of the atmosphere, while the third, which will generally be relatively small at greater heights, depends on the moisture released from the source. Moreover, equation (10) shows that all plumes from dry sources will have a humidity distribution which is independent of the source strength.

In order to find the level at which the plume becomes saturated it will be assumed, for the time being, that there is no dynamical effect due to condensation; a subsequent correction can be applied above this height if further information is required. The height at which saturation will be attained can now be determined graphically on a tephigram by plotting relation (10) for specific humidity together with the corresponding solution for the potential temperature of the plume air,

$$\theta = \theta_0 + \frac{5F}{6\pi\alpha\beta g}\left(\frac{9\alpha F}{10\pi}\right)^{-1/3} x^{-5/3}. \tag{11}$$

This graphical determination of saturation heights is illustrated in figure 1 for a range of values of heat output ($Q = \rho_1 c_p F/\beta g = 0$, $2{\cdot}5 \times 10^3$, 4×10^4, $1{\cdot}5 \times 10^5$ cals per sec); each source is assumed to be dry (i.e. $E = 0$). Each of the heavily drawn curves (marked with its appropriate source strength) shows the variation in temperature of air rising in the buoyant plume up to the level at which saturation is reached, where it is terminated at the heavy broken line which represents the dew point for plume air from dry sources of all strengths (see equation (10)). The limiting case of zero source strength represents the hypothetical ascent of a plume devoid of buoyancy, and it serves to define a lower level below which cloud formation cannot occur under the specified conditions. For purposes of reference, a second heavy broken line (to the left of the other curves) has been drawn to show the dew point of the environment, which is at uniform potential temperature.

In figure 1 the scale of absolute temperature ($T°$ C) is shown by vertical lines, that of potential temperature ($\theta°$ C) by horizontal lines, that of pressure (p millibars) by continuous diagonal lines sloping to the right and parallel with the ground level, and that of the saturation specific humidity (q_s gm per kgm) by broken diagonal lines sloping to the left. It should be noted that the specific humidity used in the analysis has measured the mass of water vapour in unit mass of moist air, in contrast with the usual meteorological unit of mass per kgm of moist air. For the purposes of the calculation the ambient air is assumed to be at uniform potential temperature, the temperature at the ground is taken as 20° C, and the profile of specific humidity in the atmosphere is represented by relation (9) with the constants $A = 11{\cdot}2 \times 10^{-3}$ gm per gm and $B = 3{\cdot}5 \times 10^{-8}$ gm per gm per cm.

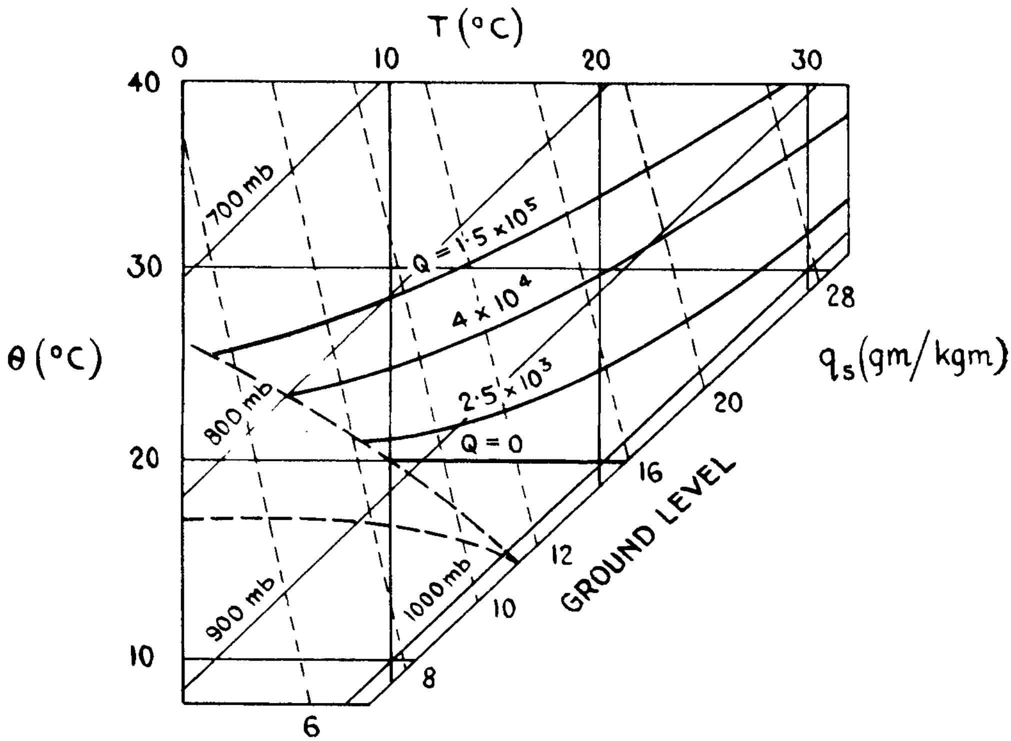

Figure 1. The ascent of thermal plumes in a moist, neutrally stable atmosphere, represented on a tephigram. The continuous heavy curves show the temperature of plume air above a range of source strengths (Q); they are terminated at the saturation level on a broken line showing the dew-point in all plumes. The broken curve to the left shows the variation with height of dew-point in the environment, which is at uniform potential temperature $\theta = 20°$ C.

It may be observed from this family of curves that in a uniform atmosphere all thermal plumes will reach levels where condensation is possible, although these may lie at heights beyond the range of the approximate theory. Thus, in each case the formation of a cloud can take place if suitable nuclei are present. However, the assumption of uniform potential temperature for the atmosphere is rather too great a simplification, and the results shown in figure 1 are of value mainly because they illustrate the variation of specific humidity with height. The results of the analysis may also be used to find the distribution throughout a buoyant plume of any material that is being released from the source, or being entrained into the plume from the surrounding fluid.

Plumes in a stably stratified atmosphere

The atmosphere is commonly in a state of stable stratification: for this reason the treatment above will be extended in this section to the ascent of thermal plumes rising in a stably and uniformly stratified atmosphere. Although the method will be similar to that of the main analysis of paper I (pp. 6–10), the Gaussian profiles used there will be replaced now by 'top hat' profiles. This modification is necessary because two attempts to impose the normal profile in a treatment of buoyant plumes (Priestly & Ball 1955, and paper I) have led to errors in the solutions for the regions of negative buoyancy. Moreover, although the normal distribution profile probably provides a reasonable time-average representation of the horizontal variation of vertical velocity and excess temperature relative to fixed axes, the mean profile relative to the plume axis is likely to be more nearly rectangular in form and of greater significance in finding the level at which the plume air saturates.

It will be assumed, therefore, that the mean vertical velocity, excess temperature and excess specific humidity are constant across the plume and zero outside it, and can be represented by 'top hat' profiles for the approximate treatment of the present note. In deriving the equations it will be sufficient, for the present purposes, to make use of the previous approximations (see paper I, equations (7)) and, in addition, to ignore the effects of water vapour on the properties of air up to the level at which condensation begins. The relations representing conservation of mass, momentum, heat, and water vapour can then be written

$$\left.\begin{aligned} \frac{d}{dx}(b^2u) &= 2\alpha bu, \\ \frac{d}{dx}(b^2u^2) &= \beta g b^2(\theta-\theta_0), \\ \frac{d}{dx}[\beta g b^2 u(\theta-\theta_0)] &= -\beta g b^2 u\frac{d\theta}{dx}, \\ \frac{d}{dx}\left(b^2u\frac{q-q_0}{q_1}\right) &= -b^2u\frac{1}{q_1}\frac{dq_0}{dx}, \end{aligned}\right\} \tag{12}$$

where $q_1 = q_0(0)$ provides a reference for q. To find the general nature of the convective motion, it is sufficient to restrict attention to the case in which both the potential temperature and the specific humidity in the undisturbed atmosphere vary linearly with height. The environment is then characterized by the two parameters

$$G = \beta g\, d\theta_0/dx \quad \text{and} \quad I = -(1/q_1)\, dq_0/dx.$$

As before, select the new variables

$$V = bu, \qquad W = b^2u, \qquad F^* = \frac{F}{\pi} = \beta g b^2 u(\theta - \theta_0),$$

and

$$M^* = \frac{M}{\pi} = b^2 u \frac{q - q_0}{q_1} \tag{13}$$

(in the present case F is the flux of buoyancy and M is related to the flux of water vapour in the plume; $\rho_1 c_p F_0/\beta g$ and $\rho_1 q_1 M_0$ may be taken as the fluxes of heat and water vapour, respectively, from the source). Equations (12) can then be written in the simplified form,

$$\frac{dW}{dx} = 2\alpha V, \qquad \frac{dV^4}{dx} = 2F^*W, \qquad \frac{dF^*}{dx} = -GW, \qquad \frac{dM^*}{dx} = IW. \tag{14}$$

In thermal plumes of the type considered, there is zero flux of mass and momentum from the source, which is taken as a point, and a specified flux of heat and moisture. Therefore, the boundary conditions at $x = 0$ can be written

$$V = 0, \qquad W = 0, \qquad F = F_0, \qquad M = M_0. \tag{15}$$

Equations (14) and the conditions (15) are reduced to their simplest non-dimensional form by the transformations

$$\left.\begin{aligned} x &= 2^{-5/8}\pi^{-1/4}\alpha^{-1/2}F_0^{1/4}G^{-3/8}x_1, \\ V &= 2^{1/4}\pi^{-1/2}F_0^{1/2}G^{-1/4}v, \\ W &= 2^{5/8}\pi^{-3/4}\alpha^{1/2}F_0^{3/4}G^{-5/8}w, \\ F^* &= \pi^{-1}F_0\, f, \\ M^* &= \pi^{-1}IF_0\, G^{-1}m. \end{aligned}\right\} \tag{16}$$

It follows that the most general form of the equations representing the ascent of thermal plumes in a uniformly stratified atmosphere is

$$\frac{dw}{dx_1} = v, \qquad \frac{dv^4}{dx_1} = fw, \qquad \frac{df}{dx_1} = -w, \qquad \frac{dm}{dx_1} = w; \tag{17}$$

and the solutions to these equations must satisfy, at $x_1 = 0$, the boundary conditions

$$v = 0, \qquad w = 0, \qquad f = 1, \qquad m = \frac{GM_0}{IF_0} = D, \tag{18}$$

where D is a dimensionless parameter. In spite of the changes that have been made in equations (12) and relations (16) owing to the introduction of the 'top hat' profile, the first three equations of (17) and first three conditions of (18) are precisely the same as the equivalent groups in paper I. Hence, the results of the numerical integration performed in paper I can be carried over directly to the present case. Finally, a solution is required for the last equation of (17) satisfying the condition $m = D$ at $x_1 = 0$. The solution for f can now be used to infer that for m, and a set of values for m is shown in table 1 for the particular case of a dry source (i.e. $D = 0$). To obtain the equivalent solution for a source emitting water vapour, the relevant value of D should be added to each value in table 1.

All dry turbulent thermal plumes are geometrically similar, although the scale of the convective motion depends on the parameters F_0 and G. In contrast, the effect of releasing moisture from a source in a damp atmosphere depends on F_0, G, I and M_0; and hence $D = GM_0/IF_0$ may be regarded as a non-dimensional parameter which represents the distribution of moisture in a thermal plume (or of the equivalent quantity in some corresponding system). A set of solutions for chosen values of the

x_1	m	x_1	m	x_1	m
0	0	1·0	0·1364	2·0	0·8487
0·1	0·0003	1·1	0·1757	2·1	0·9632
0·2	0·0019	1·2	0·2213	2·2	1·0860
0·3	0·0055	1·3	0·2736	2·3	1·2173
0·4	0·0199	1·4	0·3328	2·4	1·3568
0·5	0·0215	1·5	0·3993	2·5	1·5045
0·6	0·0350	1·6	0·4733	2·6	1·6599
0·7	0·0528	1·7	0·5550	2·7	1·8227
0·8	0·0753	1·8	0·6448	2·8	1·9912
0·9	0·1031	1·9	0·7426		

Table 1. The numerical solution of the non-dimensional equation for the flux of moisture in a thermal plume rising above a dry source of heat in a moist atmosphere.

parameter D is shown graphically in figure 2; the non-dimensional quantity m/w plotted is proportional to the value of $(q-q_0)/q_1$ on the axis of the plume. The curve for $D = 0$ corresponds to a source emitting heat but not moisture, and should apply to many natural sources; curves with $D > 0$ will apply, for example, to the plumes rising from certain industrial sources. The following figures show orders of magnitude which might be associated with the column rising from the chimney of a large power station: ground temperature 24° C, constant decrease of temperature with height

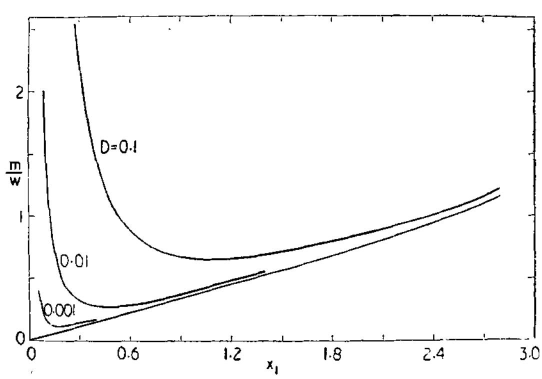

Figure 2. The variation of excess specific humidity (proportional to m/w) with height (proportional to x_1) for a thermal plume rising in a stably stratified atmosphere; the various values of the dimensionless parameter D correspond to different rates of emission of water vapour from the source.

in the atmosphere 6·5° C per km, specific humidity at the ground 10 gm per kgm, gradient of specific humidity $-$ 3gm per kgm per km, heat output 5×10^5 kW, water output (assuming half the heat wasted, and that 0·15 gm of water is produced in burning 1 gm of coal) $4{\cdot}5 \times 10^3$ gm per sec. Under these conditions $D = 3 \times 10^{-3}$, although this value may be increased by water vapour drawn into the plume from the heat exchanger towers.

To find the approximate height at which the air in the plume first saturates it is necessary to consider particular cases. Figure 3, which is also plotted on a tephigram, shows a set of curves relating the temperature and height for the range of rates of heat output $Q = \rho_1 c_p F_0/\beta g = 4{\cdot}1 \times 10^6$, $3{\cdot}3 \times 10^6$, 1×10^6, 2×10^5, and $1{\cdot}3 \times 10^4$ kW from sources which emit no moisture (i.e. $D = 0$). Each plume is represented by a continuous curve corresponding with measurements of temperature and a broken curve showing the dew-point for air rising in the plume. In the lower parts of a plume there is little effect due to stratification of the ambient air, so that these dew-point curves show individual behaviour depending on the source strength only in the uppermost part of each plume; hence, the lower part of each broken curve coincides with that drawn in figure 1 for the case of an unstratified environment, and these parts overlap in the diagram. The dew-point curve for the environment has again been drawn (on the extreme left). Where the temperature and dew-point curves intersect, the plume air becomes saturated with water vapour, although it can be seen that this happens only with the stronger sources ($Q \geqslant 4{\cdot}1 \times 10^6$ kW); when they do not intersect, the two curves for a particular plume can be identified as they terminate at the same height. Conditions in the atmosphere have been taken as follows: ground temperature 20° C, ground specific humidity 11·2 gm per kgm, $G = 1{\cdot}09 \times 10^{-4}$ c.g.s. units and $I = 3{\cdot}12 \times 10^{-6}$ c.g.s. units (i.e. a decrease of 6·5° C and 3·5 gm per kgm in a kilometre of height). In determining these curves it has been assumed that air moved vertically undergoes dry adiabatic changes at all levels in the plume, so that no account has been taken of condensation and the consequent release of latent heat above the critical levels. It can be seen from figure 3 that for given atmospheric conditions the air in a thermal plume will ultimately saturate, provided that the source is sufficiently strong; these results give no information about conditions above the saturation level. When the source emits additional moisture the air will saturate at a lower level, although it can be inferred from figure 2 that this reduction in height will generally be small.

The saturation levels predicted from these calculations depend on the use of 'top hat' profiles, and it can be shown that they are too high generally. If the excess temperature and excess specific humidity are actually greatest near the axis of the plume, the air will saturate first in this neighbourhood at some distance below the predicted height. Subsequently, as the plume air rises, saturation will spread outwards from the axis; but it is difficult to estimate these effects adequately without a knowledge of the actual mean profiles of temperature and specific humidity in atmospheric plumes. It may also be remarked at this stage that some relatively poor approximations have been used in the numerical examples; however, these are intended primarily as realistic illustrations of the model, and they are sufficient to show how to deal with particular cases of convection in the atmosphere.

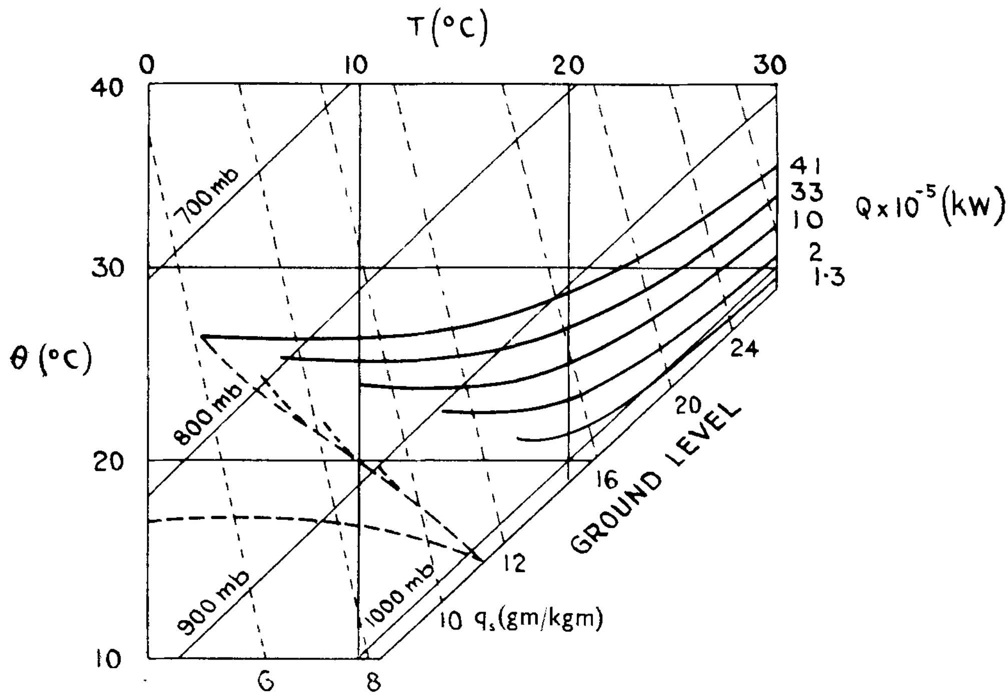

Figure 3. A tephigram showing the behaviour of a family of plumes from sources of different strengths. For each source strength, a heavy continuous curve shows the plume temperature, and a broken curve the dew-point for plume air (these dew-point curves coincide except in the upper parts of each plume). Saturation occurs only when these two curves intersect below the height of the plume top. The dew-point curve for the environment is again shown to the left.

Cloud formation over cane fires

There is some difficulty in choosing a simple application for the calculations of the previous section, as the necessary observations are seldom all available. However, much of the important information has been obtained for one case, where cloud formation is known to be associated with thermal plumes. It is apparently the practice to run a fire through the fields of sugar cane in Hawaii before harvesting, and growing cumulus clouds are often observed over the plumes of smoke from these fires. The following information has been provided by W. A. Mordy of the Pineapple Research Institute at Hawaii.

In a typical cane fire on a Hawaiian sugar plantation, 4×10^4 square metres of cane is burnt in about half an hour. On a field bearing 9×10^4 kgm of cane per acre, $1{\cdot}8 \times 10^4$ kgm of leaves will burn in the fire. Half of this is water and will evaporate, and a further $4{\cdot}5 \times 10^3$ kgm of water will be formed by combustion, so that in all $1{\cdot}35 \times 10^4$ kgm of water will be liberated per acre during the fire. Heat is also generated at the rate of $4{\cdot}4 \times 10^6$ cals per kgm when moisture-free cane fibre burns, corresponding with $2{\cdot}2 \times 10^6$ cals per kgm of the actual leaf material. Thus, during the half-hour of firing, water vapour will be released at the rate of about 75 kgm per sec and heat at the rate of about $9{\cdot}4 \times 10^5$ kW. In addition, figure 4 shows an atmospheric sounding taken in the early part of a morning on which cane-fire clouds were observed over Oahu. If no account is taken of conditions very close to the ground, the various parameters may be given the following approximate numerical values (in c.g.s. units) to represent the ascent of the thermal plume above such a burning cane field: $F_0 = 2{\cdot}75 \times 10^{12}$, $G = 1{\cdot}09 \times 10^{-4}$, $I = 2{\cdot}56 \times 10^{-6}$, $M_0 = 1{\cdot}56 \times 10^9$ and $D = 0{\cdot}025$.

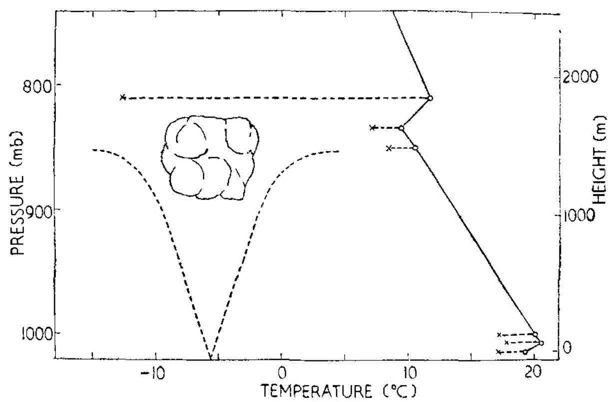

Figure 4. The growth of plume clouds above sugar plantation fires at Oahu, Hawaii. An atmospheric sounding is shown with measurements of temperature (θ) and dew-point (X), and also a diagrammatic representation of the plume and cloud.

Before applying the analysis of the previous section, it will be necessary to make some allowance for the fact that a considerable area of the field will be burning at any particular time. Although the determination of virtual sources has already been considered in paper I, there are several additional difficulties here since the moisture content of the air near the source must be known if a saturation level is to be predicted. The effective size of the fire is unknown in this case, and it is difficult to estimate an appropriate area for the ground source because of the inflow of air near the ground and the emission of combustion gases. Table 2 shows a set of calculated results for the heights at which saturation occurs and to which the plume-top would rise if there were no condensation. The corresponding area of the source at ground level is also given for each of the virtual sources considered. These results illustrate the general effect on the motion of the plume due to a large source; in the present case it seems likely that saturation will occur at about 900 metres. However, no observation of cloud height was reported from Oahu, so that these calculations must be regarded as providing an illustration of the method rather than a test of the model.

Depth of virtual source (metres)	Ground area of source (sq. metres)	Saturation height (metres)	Height of plume-top (metres)
0	0	1106	1482
53	70	1064	1429
106	280	1021	1376
159	630	984	1323
212	1120	947	1270
265	1753	910	1217
318	2530	873	1164

Table 2. The saturation level and height of the plume-top above the ground for the thermal plume over a burning cane field corresponding with several positions of the virtual source; the corresponding ground area of the source is shown in each case.

A similar calculation for a dry source (i.e. $D = 0$) of small area indicates that the corresponding saturation level will be raised by no more than a few metres. According to this approximate theory, therefore, moisture released from the burning cane has little effect in producing saturation when this occurs at heights of about a kilometre. Thus the formation of the cloud is due to the general conditions existing in the atmosphere and not to the emission of water from the source.

An approximate profile for the plume has been marked in broken lines on figure 4 for the case of a small source at the ground, and a cloud has been sketched in to indicate the appropriate condensation level. As water condenses the temperature in the cloud will rise, and the consequent increase in the buoyancy forces will cause the uppermost parts of the cloud to extend above the original top of the plume. This effect will be considered in the following section.

The growth of clouds in thermal plumes

In the previous sections an approximate theory of thermal plumes in the atmosphere has been extended to show the height at which the rising air will become saturated. Above this level it may be assumed that a cloud will develop if suitable condensation nuclei are present. For a more complete investigation of the clouds which grow in thermal plumes, it will be necessary to allow for the way in which condensation and evaporation effects vary throughout the condensed cloud, and this will require improvements in the model. However, some information can be obtained for the earlier stages of the development of such plume clouds by a simple extension of the methods used already.

For this purpose it will be assumed that there is a steady (or slowly varying) cloud in the upper regions of a thermal plume, and a modified set of conservation relations will be applied above the condensation level. Although the cloud is unlikely to occupy the full width of the plume it will be convenient to take an averaged distribution of liquid water with a rectangular profile of the same width as the plume. It will be assumed also that the condensed water remains in very small droplets which move with the neighbouring air, and that only a small proportion of the water vapour condenses; consequently, both the relative motion of the drops and plume air and the small reduction in volume caused by the condensation will be neglected.

The equations representing the conservation of mass, momentum, heat, and water (including both gaseous and liquid phases), can now be extended from their previous forms (12) to allow for the liberation of latent heat on condensation and the presence of liquid water. As the mass of liquid water per unit mass of the cloud mixture is small, they may be written approximately as

$$\left.\begin{aligned} \frac{d}{dx}(b^2u) &= 2b\alpha u, \\ \frac{d}{dx}(b^2u^2) &= b^2\beta g(\theta-\theta_0) - b^2 g\sigma, \\ \frac{d}{dx}[b^2u\beta g(\theta-\theta_0)] &= -b^2u(\beta g\, d\theta_0/dx) + \frac{\beta g L}{c_p}\frac{d}{dx}(b^2u\sigma), \\ \frac{d}{dx}(b^2uq) + \frac{d}{dx}(b^2u\sigma) &= 2b\alpha u q_0, \end{aligned}\right\} \tag{19}$$

where $\rho\sigma$ is the mass of liquid water droplets per unit volume, L is the latent heat of vaporization for water, and c_p is the specific heat at constant pressure for air. The variation of absolute temperature in small clouds will be small, so that L and c_p can be taken as constants for the present purposes.

One more relation is required before a solution can be obtained, and this represents the condition for saturation within the plume; it can be written in the form

$$q_s = q_s(x, \theta), \tag{20}$$

where q_s is the specific humidity required to saturate air at potential temperature θ and height x. It has already been pointed out that the variation of temperature within the cloud will be small provided that the treatment is restricted to the early stages of cloud development; under these conditions q_s can be expressed by a simple approximate algebraic form.

Just as equations (12) were reduced to the dimensionless set (17), the same transformations, with the additional relations

$$W\sigma = K \quad \text{and} \quad K = \pi^{-1}g^{-1}F_0 k,$$

can be used now to reduce equations (19) to dimensionless form:

$$\left.\begin{aligned} \frac{dw}{dx_1} &= v, \\ \frac{dv^4}{dx_1} &= (f-k)w, \\ \frac{df}{dx_1} &= -w+n\frac{dk}{dx_1}, \\ \frac{dm}{dx_1}+N\frac{dk}{dx_1} &= w, \end{aligned}\right\} \tag{21}$$

where $n = \beta L/c_p$ and $N = G/gq_1 I$ are two additional non-dimensional parameters needed to specify the cloud system. The first depends only on the physical constants of air and water and will vary little from one plume cloud to another, while the second parameter provides a measure for the relative stratification of temperature (or of density) and moisture.

In order to obtain a suitable form of equation (20) for use in numerical calculation, it will be assumed that the saturation water-vapour pressure is linearly related with the temperature over the range of a few degrees centigrade which may occur within the plume cloud. Although such a relationship is not strictly correct, it will be sufficient for the present approximate treatment. When due allowance is made for the variation of pressure with height, equation (20) can be written as a relation between q_s and θ:

$$(q_s-q_0)/q_1 = X_1(x)(\theta-\theta_0)+X_2(x),$$

in which the coefficients X_1 and X_2 are functions of the height x, and will depend on the particular case considered. Under the transformations used already this reduces to

$$m = X_1(x_1)f+X_2(x_1)w, \tag{22}$$

where X_1 and X_2 are appropriate functions.

While a numerical solution can be found without difficulty for equations (21) and (22) in any particular case, a general solution cannot be given because of the occurrence of two unknown parameters (n and N) in equations (21) and the slight variations of the coefficient functions in equations (22), and also because the boundary conditions at the cloud base are very sensitive to the height at which saturation occurs first. These boundary conditions require that there must be continuity in the flux of mass, momentum, heat, and water vapour, across the saturation level, and no flux of liquid water. When these are transformed into conditions on w, v, f, m and k, it is apparent that the nature of the solutions will depend very much on the height x_1 at which saturation occurs first in the plume (see table 1, paper I).

There are two kinds of behaviour in the plume cloud, and to illustrate these it is necessary to describe the results of two calculations. In each case it will be assumed that there is a thermal plume rising above a source of small area and strength 10^6 kW in an atmosphere with ground temperature

20° C and a uniform decrease with height of 6·5° C per km. In the first case the specific humidity will be taken as 12·23 gm per kgm at the ground, and with a constant rate of decrease with height of 3·5 gm per kgm per km. It may be shown that saturation occurs at $x_1 = 2{\cdot}5$ (i.e. at a height of about 1320 metres); above this level, a numerical analysis must be performed in order to obtain a new solution for equations (21) and (22). This may be described conveniently in terms of the following non-dimensional ratios: w/v, which is proportional to the plume radius b; v^2/w, which is proportional to the vertical velocity u in the plume; f/w, which is proportional to the excess of temperature $(\theta-\theta_0)$; m/w, which is proportional to the excess of specific humidity $(q-q_0)$; and k/w, which is proportional to the concentration σ of liquid water in the plume cloud. The physical variables can then be recovered from the relations

$$\left.\begin{aligned} x &= 1{\cdot}341 F_0^{1/4} G^{-3/8} x_1, \\ b &= 0{\cdot}354 F_0^{1/4} G^{-3/8} w/v, \\ u &= 1{\cdot}896 F_0^{1/4} G^{1/8} v^2/w, \\ \beta g(\theta-\theta_0) &= 1{\cdot}341 F_0^{1/4} G^{5/8} f/w, \\ (q-q_0)/q_1 &= 1{\cdot}341 F_0^{1/4} G^{-3/8} Im/w, \\ g\sigma &= 1{\cdot}341 F_0^{1/4} G^{5/8} k/w. \end{aligned}\right\} \tag{23}$$

Figure 5 illustrates the case of a plume in which saturation occurs at the level corresponding with $x_1 = 2{\cdot}5$. The continuous curves show the solutions for equations (17) below $x_1 = 2{\cdot}5$, and for equations (21) and (22) above that level. In order that the effects of condensation may be shown clearly, the solutions to equations (17) (i.e. for the unsaturated plume) are continued above the saturation level by means of broken lines. Although the saturation level is some 160 metres below the top of the undisturbed plume, it can be seen that the only effects due to the presence of the plume cloud are that the cloud top rises an extra forty metres, and that the internal temperature increases slowly with height (instead of decreasing) although the air inside the cloud still remains colder than its surroundings.

The second case, illustrated in figure 6, is for a plume of the same strength ascending in an atmosphere which is similar to that of the previous case except that the ground specific humidity is now taken as 12·84 gm per kgm; the corresponding saturation level is $x_1 = 2{\cdot}1$ (a height of about 1110 metres). It can be seen that in this case the cloud will dominate the behaviour of the upper part of the plume. Within the plume cloud the excess temperature (and hence the buoyancy) increases with height, so that the cloud extends upwards until its uppermost parts reach an inversion strong enough to prevent further ascent.

It is now possible to make certain general statements about plume clouds. If saturation occurs first near the top of the plume a small cloud can exist in the uppermost part of the plume without producing very much effect on the air flow. For lower saturation levels almost the only effect is that the steady clouds will be correspondingly taller; in all these cases

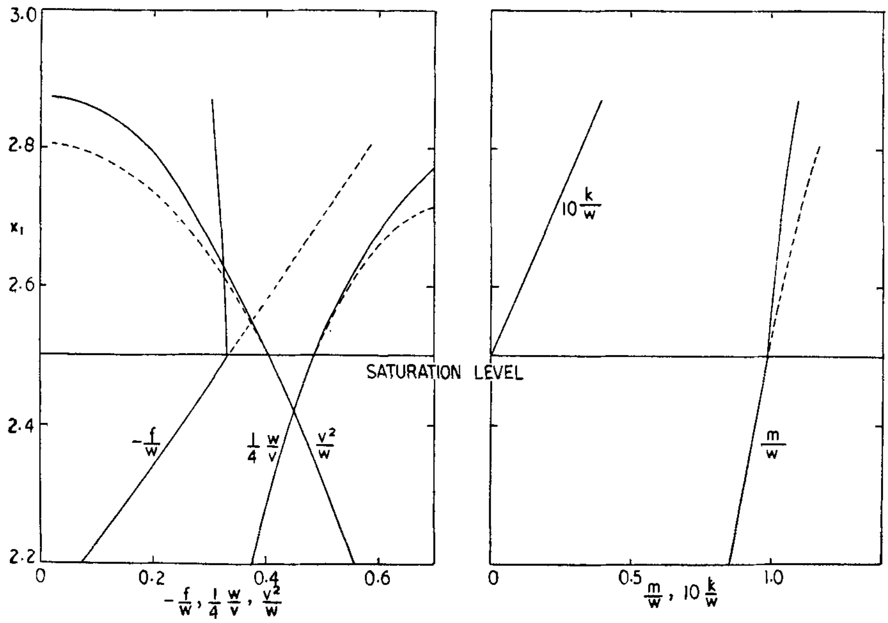

Figure 5. The ascent of a thermal plume for which the saturation level corresponds to $x_1 = 2{\cdot}5$. Below this level the curves illustrate the ascent of an unsaturated plume, while above the saturation level these solutions are continued as broken curves and the actual behaviour of the air in the cloud is shown by continuous curves. The non-dimensional variables plotted are : $w/v \propto$ plume radius; $v^2/w \propto$ vertical velocity; $f/w \propto$ excess temperature; $m/w \propto$ excess specific humidity; and $k/w \propto$ liquid water concentration within the plume.

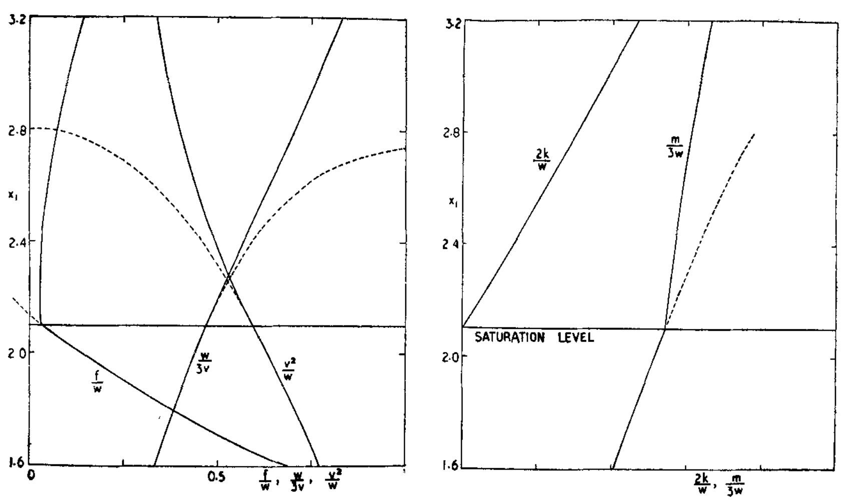

Figure 6. The ascent of a thermal plume for which the saturation level corresponds to $x_1 = 2{\cdot}1$. The variables plotted are as in figure 5.

the clouds may be regarded as controlled by the plumes. However, when the saturation level occurs below a critical height, which appears to be a little above $x_1 = 2{\cdot}2$ (i.e. $0{\cdot}03 F_0^{1/4} G^{-3/8}$ metres), the behaviour of the flow in the upper part of the plume is changed completely. The vertical extent of the plume cloud is no longer determined by the parameters characterizing the plume, but must depend on factors which have been omitted in the present simple model. These will include variations in the structure of the environment, such as strong inversions, and possibly the more complicated pattern of flow which will occur in real clouds. It may be noted in passing that this prediction of a critical size for clouds may also be of importance when the clouds are formed in other ways, for example in lee waves.

The present treatment has been developed for a steady (mean) state, and it cannot be used directly to provide information about the ascent of plumes above sources of steadily increasing strength, and about the consequent process of cloud growth that may take place. However, the conclusions can be applied to give a qualitative idea of what may happen in slowly varying systems. In addition, while the results obtained should be reasonable for clouds which are controlled by their parent plumes, they are unlikely to be satisfactory for clouds which dominate the upper parts of their plumes. For a description of these larger clouds, it will almost certainly be necessary to make more detailed allowance for the liberation and absorption of latent heat. It is clear that the liberation of heat in the lower and inner parts of this section of the plume, and the absorption in the upper and outer parts, will cause a considerable modification of the temperature distribution which has been assumed. There will be a tendency to increase the upward plume velocity at the bottom of the cloud and in the centre, and to reduce it at the top and around the sides. When a cloud grows past the critical size it may be taken that the increased buoyancy forces in the centre and the decreased forces at the edge dominate the effects due to the plume. The cloud is then no longer properly a part of the plume, and some asymmetry in its structure or some wind effect may cause it to drift clear, or it may persist when the plume dies down. From the need for continuity, it is apparent that the central upwards motion must now be balanced by downdrafts at the sides, and though the motion in the cloud is turbulent it will possess to some degree the elements of a spherical vortex (not all of which will be marked by condensed water droplets). A completely new model will be necessary in order to investigate the behaviour of these developed clouds.

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